CHAPTER 1

UNDERSTANDING MATHEMATICS

This area of learning [mathematical development] includes counting, sorting, matching, seeking patterns, making connections, recognising relationships ...
Foundation Stage Curriculum (QCA, 2000: 68)

AN INSIGHTFUL CONVERSATION

Her teacher thought that 6-year-old Gemma had a good understanding of the equals sign. Gemma had no problem with sums like 2 + 3 = □ and even 8 + □ = 9. Then the teacher asked her how she did 2 + □ = 6. Gemma replied, 'I said to myself, two, (then counting on her fingers) three, four, five, six, and so the answer is four. Sometimes I do them the other way round, but it doesn't make any difference.' She pointed to 1 + □ = 10. 'For this one I did ten and one, and that's eleven.' She pointed to 1 + □ = 10. 'For this one I did ten and one, and that's eleven.'

This conversation prompts us to ask the following questions:

- How does Gemma show here that she has some understanding of the concept of addition?
- What about her understanding of the concept represented by the equals sign?
- How would you analyse the misunderstanding shown at the end of this conversation?

In this chapter

In this chapter we discuss the importance of teaching mathematics in a way that promotes understanding. So we aim to help the reader understand what constitutes understanding in mathematics. Our main theme is that understanding involves establishing connections. For young children learning about number, connections often have to made between four key components of children's experience of doing mathematics: symbols, pictures, concrete situations and language. We also introduce two other key aspects of understanding that will run through this book: equivalence and transformation.

Learning and teaching mathematics with understanding

This book is about *understanding* mathematics. The example given above of Gemma doing some written mathematics was provided by a Key Stage 1 teacher in one of our groups. It illustrates some key ideas about understanding. First, we can recognize that Gemma does show some degree of understanding of addition, because she makes connections between the symbol for addition and the process of counting on, using her fingers. We discuss later the particular difficulties of understanding the equals sign that are illustrated by Gemma's response towards the end of this conversation. But we note here that, as seems to be the case for many children, she appears at this point to perceive the numerical task as a matter of moving symbols around, apparently at random and using an arbitrary collection of rules.

Learning with understanding

Of course, mathematics does involve the manipulation of symbols. But the learning of recipes for manipulating symbols in order to answer various types of questions is not the basis of understanding in mathematics. All our experience and what we learn from research indicate that learning based on understanding is more enduring, more psychologically satisfying and more useful in practice than learning based mainly on the rehearsal of recipes and routines low in meaningfulness.

For a teacher committed to promoting understanding in their children's learning of mathematics, the challenge is to identify the most significant ways of thinking mathematically that are characteristic of understanding in this subject. These are the key cognitive processes by means of which learners organize and internalize the information they receive from the external world and construct meaning. We shall see that this involves exploring the relationship between mathematical symbols and the other components of children's experience of mathematics, such as formal mathematical and everyday language, concrete or real-life situations, and various kinds of pictures. To help in this we will offer a framework for discussing children's understanding of number and number operations. This framework is based on the principle that the development of understanding involves building up connections in the mind of the learner. Two other key processes that contribute to children learning mathematics with understanding are equivalence and transformation. These processes also enable children to organize and make sense of their observations and their practical engagement with mathematical objects and symbols. These two fundamental processes are what children engage in when they recognize what is the same about a number of mathematical objects (equivalence) and what is different or what has changed (transformation).

Teaching with understanding

This book has arisen from an attempt to help teachers to understand some of the mathematical ideas that children handle in the early years of schooling. It is based on our experience that many teachers and trainees in nursery and primary schools are helped significantly in their teaching of mathematics by a shift in their perception of the subject away from the learning of a collection of recipes and rules towards the development of understanding of mathematical concepts, principles and processes. So our emphasis on understanding applies not just to children learning, but also to teachers teaching: in two senses. First, it is important that teachers of young children teach mathematics in a way that promotes understanding, that helps children to make key connections, and that recognizes opportunities to develop key processes such as forming equivalences and identifying transformations. Second, in order to be able to do this the teachers must themselves understand clearly the mathematical concepts, principles and processes they are teaching. Our experience with teachers suggests that engaging seriously with the structure of mathematical ideas in terms of how children come to understand them is often the way in which teachers' own understanding of the mathematics they teach is enhanced and strengthened.

Concrete materials, symbols, language and pictures

When children are engaged in mathematical activity, as in the example above, they are involved in manipulating one or more of these four key components of mathematical experience: concrete materials, symbols, language and pictures.

First, they manipulate *concrete materials*. We use this term to refer to any kind of real, physical materials, structured or unstructured, that children might use to help them perform mathematical operations or to enable them to construct mathematical concepts. Examples of concrete materials would be blocks, various sets of objects and toys, rods, counters, fingers and coins.

Second, they manipulate *symbols*: selecting and arranging cards with numerals written on them; making marks representing numbers on pieces of paper and arranging them in various ways; copying exercises from a work card or a textbook; numbering the questions; breaking up numbers into tens and units; writing numerals in boxes; underlining the answer; pressing buttons on their calculator; and so on.

Third, they manipulate *language*: reading instructions from work cards or textbooks; making sentences incorporating specific mathematical words; processing the teacher's instructions; interpreting word problems; saying out loud the words that go with their recording; discussing their choices with the teacher and other pupils; and so on.

Finally, they manipulate *pictures*: drawing various kinds of number strips and number lines, set diagrams, arrow pictures and graphs.

An example in a nursery class

In a nursery class some children aged 3 to 4 years are propelling themselves around the playground on tricycles. The tricycles are numbered from 1 to 9. At the end of the time for free play they put the tricycles away in a parking bay, where the numerals from 1 to 9 are written on the paving stones, matching their tricycle to the appropriate numbered position in the bay. There are conversations prompted by the teacher about why a particular tricycle is in the wrong place and which one should go next to which other ones. When all the tricycles are in place the children check them by counting from 1 to 9, pointing at each tricycle in turn (see Photograph 1.1).

We begin to see here how understanding of elementary mathematical ideas develop, as children begin to make connections between real objects, symbols, language and pictures. The children are making connections between the ordering of the numerical symbols and the ordering of the actual cars. The numerals on the paving stones form an elementary picture of part of a number line, providing a visual image to connect with the language of counting. Already the children are beginning to understand what we will call (in Chapter 2) the ordinal aspect of number, by making these simple connections between real objects lined up in order, the picture of the number line, the symbols for numbers and the associated language or counting.

Understanding as making connections

A simple model that enables us to talk about understanding in mathematics is to view the growth of understanding as the building up of cognitive connections. More specifically, when we encounter some new experience there is a sense in which we understand it if we can connect it to previous experiences or, better, to a network of previously connected experiences. In this model we propose that the more strongly connected the experience is, the greater and more secure is our understanding of it. Using this model, the teacher's role in developing understanding is, then, to help the child to build up connections between new experiences and previous learning. Learning without making connections is what we would call learning by rote.

Connections between the four key components

We find it very helpful to think of understanding the concepts of number and number-operations (that is, number, place value, addition, subtraction, multiplication, division, equals, number patterns and relationships, and so on) as involving the building up of a network of cognitive connections between the four types of experience of mathematics that we have identified above: concrete experiences, symbols, language and pictures. Any one of the arrows in Figure 1.1 represents a possible connection between experiences that might form part of the understanding of a mathematical concept.

So, for example, when a 3-year-old counts out loud as they climb the steps on the playground slide or when they stamp along a line of paving stones, they are connecting the language of number with a concrete and physical experience. Later they will be able to connect this experience and language with the picture of a number strip. When the 4-year-old plays a simple board game they are connecting a number symbol on a die with the name of the numeral and the concrete experience of moving their counter forward that number of places along the board. And so, through these connections, understanding of number is being developed.

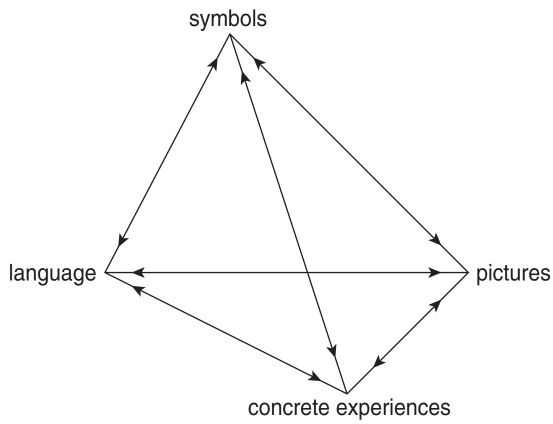

Figure 1.1 *Significant connections in understanding number and number operations*

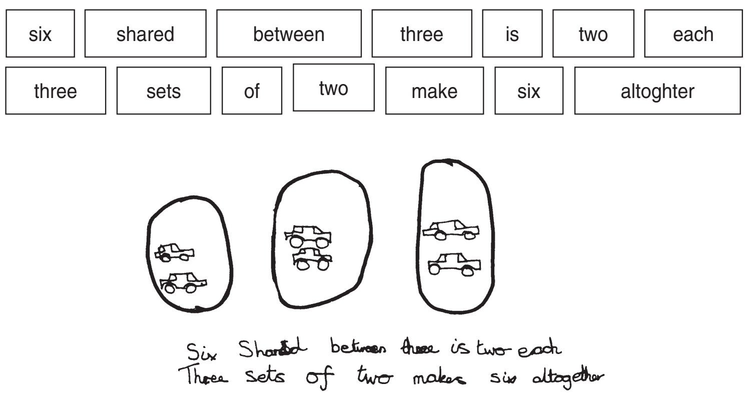

Figure 1.2 *Language patterns and a picture for a sharing experience*

An illustration: 7-year-olds and the concept of division

Below is one teacher's description of some children in her class engaged in a mathematical activity designed to develop their understanding of the concept of division. The emphasis on making connections is clearly what we would recognize as developing understanding, as opposed to just the processes of handling division calculations. The children's recording is shown in Figure 1.2. This illustrates how the activity involves children in handling the four key components of mathematical experience – real objects, pictures, mathematical symbols and mathematical language – and in making connections between them.

Three 7-year-olds in my class were exploring the early ideas of division. On their table they had a box of toy cars, paper and pencil, a collection of cards with various words written on them (*shared*, *between*, *is*, *each*, *sets*, *of*, *make*, *altogether*, *two*, *three*, *six*, *nine*, *twelve*) and a calculator. Their first task was to share six cars between the three of them. They discussed the result. Then they selected various cards to make up sentences to describe what they had discovered. The children then drew pictures of their sharing and copied their two sentences underneath. One of the children then picked up the calculator and interpreted the first sentence by pressing these keys: 6 ÷ 3 =. She seemed delighted to see appear in the display a symbol representing the two cars that they each had. She then interpreted their second sentence by pressing these keys: × 3 =. As she expected, she got back to the 6 she started with. She demonstrated this to the other children who then insisted on doing it themselves. When they next recorded their calculations as 6 ÷ 3 = 2 and 2 × 3 = 6, the symbols were a record of the keys pressed on the calculator and the resulting display. Later on I will get them to include with their drawings, their sentences, their recording in symbols, and a number line showing how you can count back from 6 to 0 in jumps of 2 (see Figure 1.3).

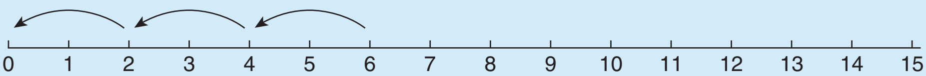

Figure 1.3 *Division connected with a number lin*

We can identify some of the connections being made by these children in this activity. They make connections between concrete experience and language when they relate their manipulation of the toy cars to the language patterns of '… shared between … is … each', and '… sets of … make … altogether'. They connect their concrete experience with a picture of three sets of two things. The language of their sentences is connected with the symbols on the keys and display of the calculator. And then, later, they will be learning to connect these symbols with a picture of three steps of two on a number line. It is because of these opportunities to make so many connections between language, concrete experience, pictures and symbols that we would recognize this as an activity promoting mathematical understanding.

We should comment here on the role of calculators in this example, since they are not normally used with children in this age range. However, here they helped pupils to connect the mathematical symbols on the keys with the concrete experiences and the language of division, showing how calculator experiences even with young children can be used selectively to promote understanding. A more detailed analysis of understanding of multiplication and division is provided in Chapter 4.

Understanding place-value notation

The connections framework outlined above is, of course, only a simple model of understanding. It is provided to enable us to discuss and recognize some significant aspects of what it means to understand mathematics. For example, let us consider understanding of place value.

The principle of place value

The principle of place value is the basis of the Hindu-Arabic number system that enables us to represent all numbers by using just ten digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9). The value that each digit represents is determined by its place (going from right to left), the first place on the right representing ones (or units), the second tens, the next hundreds, and so on, with increasing powers of ten. Thus the digit 9 in 900 represents a value ten times greater than it does in the number 90. Most teachers would agree that some understanding of place value is essential for handling numbers and calculations with confidence.

The principle of exchange

What is involved in understanding place value? What connections between symbols, language, concrete experiences and pictures might be developed and established as part of this understanding in children up to the age of about 8 years? First, there is the principle of exchange, that when you have accumulated ten in one place in a numeral these can be exchanged for one in the next place to the left, and vice versa. So ten units can be exchanged for one ten, and ten tens for one hundred, and so on. To understand this principle the child can experience it in a variety of concrete situations, learning to connect the manipulation of materials with the language pattern 'one of these is ten of those'. This might be, for example, working with base ten blocks, as shown in Figure 1.4, where a flat piece can be constructed from ten long pieces, and a long piece can be constructed from ten units.

Children would also demonstrate understanding of this principle when they reduce a collection of base ten blocks to the smallest number of pieces by a process of exchange, using the appropriate language to describe what they are doing: 'one of these is ten of those'. In our model we recognize this as an aspect of understanding because the child is making connections between language and the manipulation of concrete materials.

Connecting materials, symbols and arrow cards

To understand place value, they must also learn to connect collections of materials with the symbols, as shown in Figure 1.5. They might demonstrate understanding of

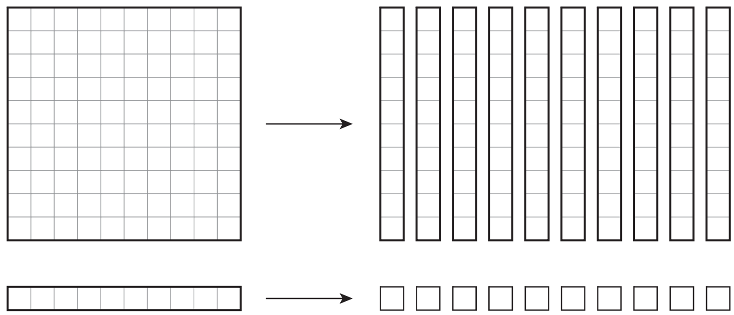

Figure 1.4 *'One of these is ten of those'*

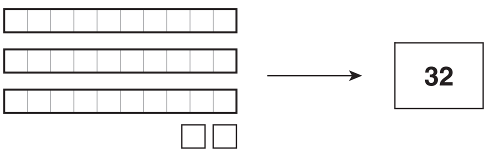

Figure 1.5 *Connecting materials and symbols*

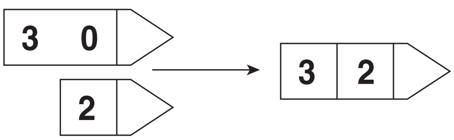

Figure 1.6 *Arrow cards to show 32 as 30 and 2*

this connection by selecting appropriate base ten blocks to correspond to any given two- or three-digit number written in symbols. Particularly helpful in establishing understanding of place value is the use of what are sometimes called arrow cards. These are illustrated in Figure 1.6. By placing the 2-card on top of the 30-card, the cards show how the symbol 32 is made up from the 30 represented by the 3 tens and the 2 represented by the 2 units.

These arrow cards therefore provide pupils with a kind of picture, an image of what is going on when we write in symbols a number with more than one digit. So, with an appropriate dialogue between the teacher and the pupil, activities with arrow cards promote connections between language, pictures and symbols.

Connecting the number names with the symbols

Understanding place value also involves learning to connect the particular ways in which we read and say out loud the names of numerals with the corresponding symbols. So, for example, the written numeral 342 is connected with 'three hundred and forty-two', spoken. The connection from the spoken form to the written form in particular is far from straightforward. Many pupils will write 'three hundred and forty-two' as 30042 or 3042, making a false connection between the word 'hundred' and the symbol for zero. This illustrates the need to develop understanding by making as many connections as possible. The child who makes this kind of error will be helped, for example, by connecting the spoken language and the written symbols with the concrete experience of blocks and the image of the arrow cards.

Zero as a place holder

One important component of understanding place-value notation is the use of zero as a place holder. Zero is the cause of many of the difficulties that pupils have in handling numbers, because the connections to be established are sometimes challenging and far from straightforward. There are particular problems in making the connections between the way we say numbers and the way we write them in symbols where zero is involved. For example, in Figure 1.7, the child must learn to connect the symbols 206 and the collection of materials shown, namely, 2 flat pieces (hundreds), no long pieces (tens) and 6 units (ones). The zero is needed to occupy the empty place, to show that there are no long pieces (tens). So the pupil has to write something to represent nothing. As we have seen above, children may wrongly connect the symbol for zero with the word 'hundred' reading 206 as two (2), hundred (0), and six (6). The difficulty is to connect the 2 with 'two hundred' on the basis of where it is written. Other aspects of understanding zero are considered in Chapters 2 and 5.

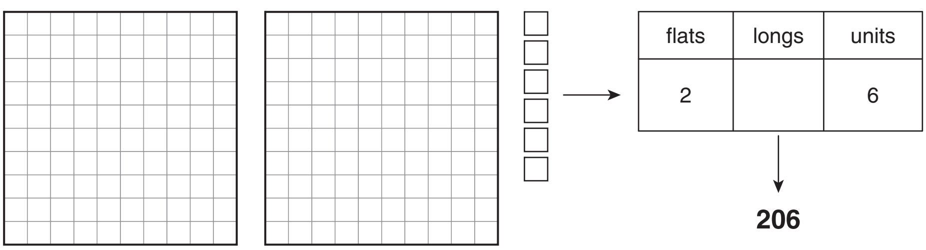

Figure 1.7 *Zero as a place holder*

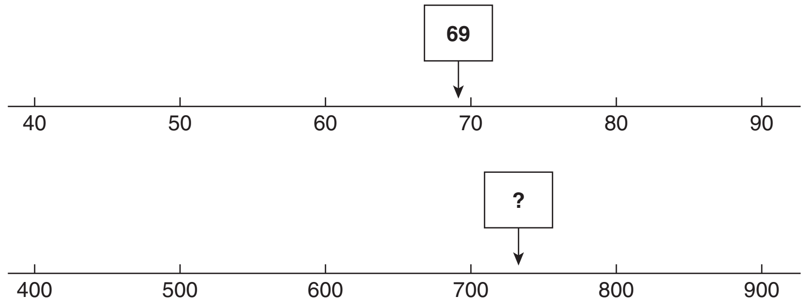

Figure 1.8 *Connecting symbols with the number line*

Connecting symbols for numbers with the number line

A further aspect of understanding of place value, connecting symbols with pictures, is shown in Figure 1.8. The child learns to connect the symbols for numbers with the important picture of the number line. This might involve, for example, locating the approximate position of a given number on a number line labelled in 10s or in 100s. Then, vice versa, the pupil has to be able to state or write the approximate number corresponding to a given point on a number line labelled in 10s or in 100s.

We can see therefore that much of what is involved in understanding place value can be identified as the building up of a complex network of connections between language, symbols, concrete materials and pictures. We find it helpful to regard many mathematical concepts as networks of such connections.

The function of a mathematical symbol

Other than just marks we learn to write on paper and manipulate according to certain rules, what are mathematical symbols? What is the function of a symbol in mathematics? What is the relationship of a mathematical symbol to our experiences of doing mathematics and handling mathematical ideas? These are some comments from some of our Foundation Stage and Key Stage 1 teachers, in response to these questions:

- I think of mathematical symbols as abbreviations. They're a sort of shorthand.
- They have very ambiguous meanings for me. They have different meanings depending on the situation you're using them in.
- They sometimes mean you have to do something. Perform an operation. Move some blocks around.

Mathematical symbols are not just abbreviations

Are mathematical symbols just abbreviations? Of course, there is a sense in which mathematical symbols (such as 4, 28, ×, =) are abbreviations for mathematical ideas or concepts. But it is important to note that this does not mean that a symbol in mathematics is just an abbreviation for a specific word or phrase. It is tempting to think of, say, the division sign as being essentially an abbreviation for the words 'shared between'. Children often appear to view mathematical symbols in this way. One 9-year-old was using a calculator to do '28 ÷ 4', saying to himself as he pressed the keys, 'Twenty-eight shared … '. At this point he turned to the teacher and asked, 'Which button's *between*?' It was as though each word had to have a button or a symbol to represent it. When we see children writing 41 for fourteen it is clear that they often say 'four' and write 4, then say 'teen' and write 1, again using the symbols as abbreviations for the sounds they are uttering. And so the same child will happily write 41 for forty-one a few lines later! It is a similar error when children record a number like three hundred and seventy-five as 30075 or 3075. The zeros are written down as abbreviations for the word 'hundred'.

But once we think of understanding, particularly understanding of number and number operations, as the building up of connections between concrete experiences, symbols, words and pictures, we begin to see that a mathematical symbol is not simply an abbreviation for just one category of concrete experiences, or just one word or phrase, or just one picture. The child has to learn to connect one symbol with what, at times, can seem to be a confusing variety of concrete situations, pictures and language.

A symbol represents a network of connections

Hence we suggest that a symbol in mathematics is a way of representing a concept, by which we mean a network of connections. The symbol then becomes a means whereby we can manipulate that concept according to various rules. Without the symbols it would be virtually impossible for us to manipulate the concepts. The symbols of mathematics allow us to both discover and express relationships between various concepts. For example, when we write down a statement in symbols like 2 + 4 = 6 we are expressing a relationship between the concepts of two, four and six, addition and equality, each of which, as we shall see, is itself a complex network of connections represented by the given symbol.

The teacher's suggestion above that mathematical symbols have different meanings depending on the situation in which they are being used is a very perceptive observation. One symbol can indeed represent a complex network of connections. It can therefore be applied to a variety of situations and pictures. And it can be associated with a variety of language. This is one of the major themes in our discussion of understanding of number and number-operations in this and subsequent chapters. We explore in considerable detail how a statement in symbols, such as 2 + 4 = 6, can be connected to an extensive range of different pictures, language and concrete situations.

The symbols for the numbers, the symbol for the operation of addition and the equals sign itself each has a variety of meanings depending on the situation and the manner in which they are being used. Put them altogether and the simple-looking statement $2 + 4 = 6$, represents a surprisingly complex network of connections. This is at one and the same time the reason why mathematics is so powerful and the reason why it is for many such a difficult subject to understand.

Consider, for example, the symbol for zero. In discussing zero as a place holder in a number like 206 we suggested that the zero represents nothing. In this case this is true. But zero on a number line does not represent 'nothing'! It is the symbol that is connected with the most important point on the line. If my bank balance is zero, then I have nothing; the zero indicates a complete absence of money in my account. But a temperature of zero is not an absence of temperature. So understanding the concept of zero involves building up a complex network of connections. The symbol 'zero' represents for us this network of connections.

Transformation and equivalence

The concept of 'equals' is another example of a complex network of connections, which we represent by the equals sign. To analyse this concept we first discuss two fundamental ideas that run right through mathematics. These are the notions of *transformation* and *equivalence*.

What is different? What is the same?

We introduce these principles with an example from everyday experience, namely, paper sizes. Figure 1.9 represents two pieces of paper, one A4 size and the other

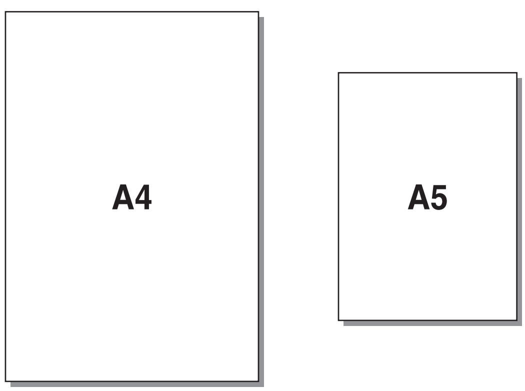

Figure 1.9 *The same but different*

A5 size. We should explain that you get A1 paper by folding A0 paper in half, A2 by folding A1 in half, and so on, and that the dimensions of the paper are cunningly chosen so that each rectangle has exactly the same proportions as the original. So the A5 rectangle is the shape produced by folding in half the A4 rectangle. Now when we begin to make mathematical statements about the relationships between the two rectangles in Figure 1.9, we find they fall into two categories. On the one hand, we may look at the rectangles and make observations about the ways in which they differ from each other:

- One rectangle is on the left and the other is on the right.
- One is half the area of the other.
- The length of one is about 1.4 times the length of the other.

Statements like these are essentially using the notion of transformation. We are concerned with the changes that are observed when we move our attention from one rectangle to the other. We are hinting at what would have to be done to one rectangle to transform it to the other. But, on the other hand, we may look at these rectangles and make statements about the ways in which they are the same. Statements like these are essentially using the notion of equivalence. We are concerned with what stays the same when we move our attention from one rectangle to the other:

- They are both rectangles.
- They are the same shape.
- Their sides are in the same proportion.

More generally, then, when we make statements about what has changed in a situation, what is different about two things, what something has become, and so on, we are using the idea of transformation. When we concern ourselves with what is the same, with similarities rather than differences, what remains unchanged in spite of the transformation, then we are talking about equivalence. The key questions in everyday language for teachers to prompt children to recognize transformations and equivalences are simply 'What is different?' and 'What is the same?'

> So when my 4-year-olds sort the shapes in the shape box into sets and put all the squares together, for example, they are seeing an equivalence. And when they look at two of the squares and say something like 'this one is bigger than that one' they are seeing a transformation.

What stays the same when things change?

Much of mathematics, not just geometrical experiences like the example above, is concerned with recognizing and applying equivalences and transformations. Often a crucial mathematical principle involves the recognition of which equivalences are preserved under which transformations. An example of this, which illustrates the point nicely, is that of equivalent fractions. The reader may recall that you can transform the fraction $^4/_6$ by dividing top and bottom by 2 to produce the equivalent fraction $^2/_3$ (and to record this as $^4/_6 = {}^2/_3$). This is a transformation that preserves the equivalence. But it is apparently not in order to transform $^4/_6$ by, say, adding one to top and bottom, because $^4/_6$ does not equal $^5/_7$. This transformation does not preserve the equivalence. But there are other situations where adding 1 to each of two numbers does preserve an equivalence, such as when calculating the difference between two numbers. For example, 77 – 49 can be correctly and usefully transformed into 78–50. At times one feels sorry for the poor pupils trying to make sense of this subject, particularly if the processes are taught as recipes and routines without understanding. If the appropriate connections are not made, it must seem entirely arbitrary as to whether a particular transformation is acceptable and warrants a tick or is unacceptable and generates a cross.

The equals sign

Finally, to bring together the major ideas about understanding introduced in this chapter, we return to the equals sign (=) and the difficulties illustrated by Gemma at the start. We can now see that the essence of the problem with this symbol is that the concept of *equals* is such a complex network of ideas and experiences. We find that there is not just one form of words that goes with the symbol (=) but that there is a range of language and situations to which the symbol may become attached, including both the ideas of transformation and equivalence. Some of the teachers with whom we worked articulated their anxieties about the meaning of this symbol.

- My 6-year-olds had problems with some questions in their maths books where they had to put in the missing numbers, like this: 6 = 2 + □. Most of them put in 8, of course. When I tried to explain to them how to do these sums I realized I didn't actually know what the equals sign meant myself. We would say 'two add something makes six' if it were written the other way round, but 'six makes two add something' doesn't make sense.
- Is it wrong to say 'four add two *makes* six'? Should I insist that the children say '*equals* six'?
- The word 'equals' doesn't mean anything to them. It's just a symbol, just some marks on paper that you make when you're doing sums.
- Doesn't it confuse children to say 'makes' when you're adding and then to say 'leaves' when you're taking away?
- And sometimes we just read it as 'is', like 'three add four is seven'.

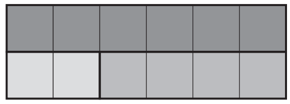

Figure 1.10 *Two add four is the same as six*

The equals sign representing an equivalence

Strictly speaking, the equals sign represents the idea of equivalence. When we write down 2 + 4 = 6 we are expressing an equivalence between 2 + 4'and 6. We are making a statement that there is something the same about 'two added to four' and 'six'. Probably the most straightforward language to go with this statement is 'two add four is the same as six'. To emphasize the underlying equivalence in statements in arithmetic that use the equals sign, this phrase 'is the same as' is particularly significant. It connects very clearly with the concrete experience of doing addition and subtraction with some structured materials as shown in Figure 1.10.

When the child makes a train with a 2-rod and a 4-rod the problem is to find another rod to match this train. Recording this experience as 2 + 4 = 6 is an expression of the equivalence: the 2-rod added to the 4-rod is in one sense the same as the 6-rod. Of course, they are only the same in that they are the same length. A train made up of a blue rod and a brown rod is very different from a red rod. But lying there side by side they represent an equivalence, and this is expressed by the symbols 2 + 4 = 6. It is worth noting that this interpretation of the equals sign makes sense of the problem that Gemma started with: '6 = 2 + □' can be read as 'six is the same as two add something'.

The equals sign representing a transformation

However, when the child puts out sets of two counters and four counters, forms their union and counts the new set to discover that there is now a set of six counters, it is a bit obscure to suggest that this is an experience of 'two add four is the same as six'. The child has actually *transformed* the two sets of two and four counters into a set of six (see Figure 1.11). The child's attention therefore is focused on the transformation that has taken place. This being so, it seems perfectly natural, and surely appropriate, to use the language 'two and four makes six' to describe the transformation the child has effected. One of the teachers quoted above said that she regarded the symbols as instructions to do something. In other words, the equals sign tells you to apply some sort of transformation. There is evidence that this is how children most frequently interpret the equals sign.

So, in practice, the equals sign represents both the equivalence and the transformation aspects of the relationship between 2 + 4 and 6. Thus we would not want to

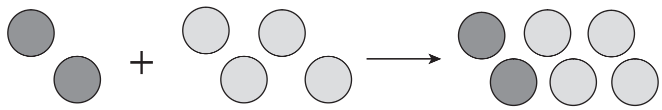

Figure 1.11 *Two add four makes six*

suggest that it is wrong or in some way mathematically incorrect to associate 'makes', 'leaves', 'is', and so on, with the equals sign, and insist on using only one particular form of words, such as 'is the same as' or even 'equals'. Rather, we would advocate a combination of experiences emphasizing the notions of both equivalence and transformation. As we have already argued, mathematical symbols are not just abbreviations for particular words or phrases. We have to recognize that the statement 2 + 4 = 6 is actually at one and the same time a representation in symbols of the transformation that has been applied to 2 and 4, and the equivalence that has emerged between 2 + 4 and 6.

One symbols, two meanings

It could be, therefore, that the child's attention might on some occasions be directed to the transformation of two and four into six, particularly when using counters, fingers, sets of toys, pencils, sweets, and so on. And on other occasions, particularly when using some structural apparatus or when making steps on a number line, the attention might be directed to the equivalence of 'two added to four' and 'six'. On both occasions the child might record their activity as 2 + 4 = 6. But this might be accompanied in the first case by the language pattern 'two add four makes six', and in the second case by the language pattern 'two add four is the same as six'. The use of different language appropriate to the situation is inevitable and perfectly acceptable, demonstrating that the child is gaining experience of both the transformation and the equivalence built into the relationship between 2 + 4 and 6. However, we should note that, as in the case of Gemma, it is often the dominant association of the equals sign with the idea of 'makes' that is the root cause of difficulties with missing number questions. Activity 1.5 at the end of this chapter provides an opportunity to explore the use of the phrase 'is the same as' in such questions.

These two ideas of transformation and equivalence are almost always present whenever we make statements of equality. In the fractions example above, when we write down $^4/_6 = {}^2/_3$ we are both recording a transformation that has been applied to the $^4/_6$ and recognizing an equivalence that has emerged. There is a sense in which $^4/_6$ is not the same as $^2/_3$. Using one meaning of the fraction notation, four slices of a cake cut into six equal parts is actually different from two slices of a cake cut into three equal parts. But there is something very significantly the same about these two situations – they produce the same amount of cake – which prompts us to recognize an equivalence and to record it with the equals sign. This is just the same with $2 + 4 = 6$. Two piles of cubes, one containing two and the other containing four, look quite different from a pile of six cubes, yet there is an aspect of sameness about the two situations that warrants the use of the equals sign.

So, what does the equals sign mean? Strictly we should concede that it means 'is equivalent to', or 'is the same as', in whatever sense is determined by the context. And we should say that perhaps this is an aspect of the meaning of the symbol that is underplayed by teachers. It would be no bad thing for children's mathematical development for the phrase 'is the same as' to occur more frequently in their talk and in the talk of their teachers. But as with all mathematical symbols we have to learn to connect the equals sign with a complex variety of situations, operations and language, sometimes focusing on the transformation and sometimes on the equivalence. Understanding mathematics right from the earliest years involves us in learning to attach the same symbol to a potentially bewildering variety of situations and language. This is even true of the first mathematical symbols we encounter, those used to represent numbers. And this will be the subject of our next chapter.

Some activities to use with children

The activities below are chosen to illustrate ways in which teachers might focus particularly on the development of understanding through making connections, in the ways discussed in this chapter.

ACTIVITY 1.1 Number of the week

Objective

To develop connections between the name for each number from zero to ten and the symbol, pictures and concrete embodiments of the number.

Materials

A card with instructions for the activity to be sent home each week with the child for the parent/guardian. Here is an example, using the number five.

The number of the week is 5

Please try to do some or all of these activities with your child this week and let us know how he or she gets on. Thank you!

- Look around for examples of sets of *five* things at home and outside in the street and ask your child to count them.
- When you are out and about ask the child to look out for the number 5 and to point it out to you.
- Tell your child that for this week they must put up *five* fingers if they want to ask you for something.
- Write down 3 2 5 1 6 8 and ask the child to point to *five*.
- Ask the child to find page *five* in a book.
- Then ask the child to find a word with *five* letters.
- If you have stairs, get the child to go up *five* stairs, counting up to *five*, and then down again, finishing with zero. Do the same thing with a row of paving stones or tiles.
- Ask the child to draw sets of *five* dots arranged in lots of different ways.
- Ask the child to draw a monster with *five* heads, *five* legs and *five* tails.

Method

Each week a particular number is chosen to be the focus. The parent/guardian is encouraged to do the activities at home with their children and to report in their home-school contact book how they get on.

ACTIVITY 1.2 Tidying the books

Objective

To develop connections between number symbols, the language of ordering numbers and putting objects in order.

Materials

A set of ten reading or picture books in a nursery class and a shelf.

Method

Compare the example of cars in the nursery playground earlier in this chapter. Number the books 1 to 10, and number the spaces on the shelf 1 to 10 (see Photograph 1.2). When the time for tidying up comes, children take it in turns to put the books back in their right places. The teacher or classroom assistant takes the opportunity to talk with the child about the ordering process. Why does this book not go here? Which number goes after number two? Does this book match that

(Continued)

(Continued)

space? Which number book is missing? What number book is that one over there? When all the books are in place ask the child to point at the books and say the numbers from left to right; when they are efficient at this, to do the same from right to left.

ACTIVITY 1.3 Bundling in tens

Objective

To provide topic-based, practical experience of the place-value principle of bundling into tens.

Materials

A Paddington Bear, pretend (or real) biscuits, packing material, and a three-minute timer with a buzzer.

Method

This could be an activity for, say, three children, as part of an ongoing project on the Paddington Bear theme. Two children package biscuits into rolls of ten at a time, while the third sets the timer going. When the buzzer sounds the third player announces that Paddington has come for his biscuits. This child, with the assistance of Paddington, counts (in tens and ones) how many biscuits each player has wrapped and awards a biscuit as a prize to the one who has packed the most. The children can take it in turns to be packers.

ACTIVITY 1.4 Connections (add and subtract)

Objective

To develop the concepts of addition and subtraction by making connections between symbols, concrete experiences, language and pictures.

Materials

A box of counters (or toys, or pennies); paper and pencil; strips of card with words associated with addition and subtraction written on them (such as: take away, add, leaves, make); cards with numbers from 1 to 20 written on them; cards with +, −, = symbols on them; a duplicated sheet of number lines marked from 0 to 20.

Method

Develop an activity, for a small group of children, using subtraction, similar to that using division outlined in this chapter. Do not provide everything outlined here at the start, but gradually introduce a wider range of experiences to be connected with subtraction. The children can be provided with a prompt sheet with these words written on it: *counters, sentences, picture,*

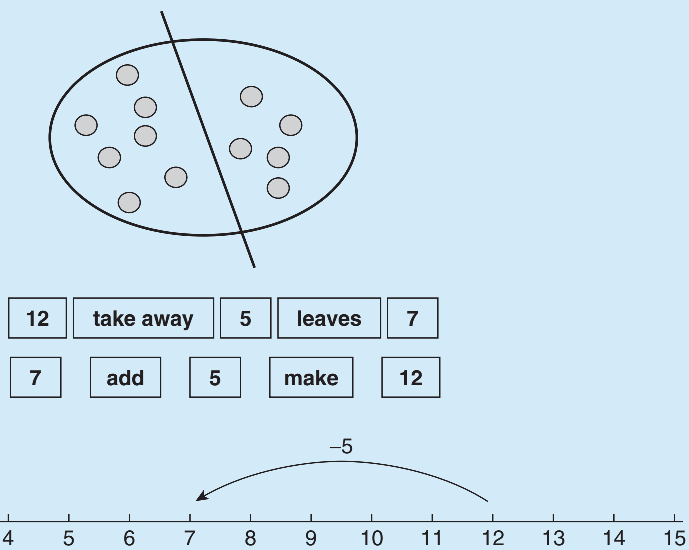

Figure 1.12 *Making subtraction connections*

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numbers, number line. The children should start with two of the numbered cards, such as 12 and 5. It is explained to them that this means they put out twelve counters, move five to one side, and count how many are left. They then have to connect this with the appropriate language. For example, they use the cards provided to make up two sentences: 'twelve take away five leaves seven' and 'seven add five make twelve'. They then have to draw a picture on their paper to show what they have done and assemble or write the sentences underneath, as shown in Figure 1.12.

To make the connection with symbols they then replace the words in the number sentences with the +, − or = cards, to produce $12 - 5 = 7$ and $7 + 5 = 12$. Finally they must represent the relationship on a number line, as shown in Figure 1.12. The teacher should discuss with the children how their number-line diagram shows both $12 - 5$, and $7 + 5$.

ACTIVITY 1.5 Missing numbers

Objective

To help children understand statements with missing numbers and the meaning of the equals sign.

Materials

Some structured materials where the numbers from 1 to 10 are represented by coloured rods or sticks of coloured cubes joined together.

Method

First investigate how the children handle a collection of missing number additions, where the box for the missing number might be in any one of six positions. For example, using the number statement $5 + 3 = 8$, they could look at: $5 + 3 = \square$, $5 + \square = 8$, $\square + 3 = 8$, $\square = 5 + 3$, $8 = \square + 3$ and $8 = 5 + \square$. Which type of questions do they find easier? More difficult? Talk with the children about how they interpret the questions, and discover what language they use for the equals sign. Explore the suggestions that children might be helped to make more sense of these questions by using the phrase 'is the same as' to go with the equals sign, and by connecting the symbols in these questions with corresponding manipulations of the materials.

ACTIVITY 1.6 Connections (place value)

Objective

To develop understanding of place value by making connections between symbols, concrete experiences, language and pictures.

Materials

A pack of cards with numerals 0 to 9 written on them, and strips of card with the words: *one, two, three, … nine, and twenty, thirty, forty, … ninety*; base-ten materials (tens and ones); 10p and 1p coins; a number line marked from 0 to 99 (such as a metre rule marked in centimetres); a movable pointer for indicating numbers on the number line; arrow cards for tens and units.

Method

This is a co-operative activity for a small group of children. They turn over two of the numeral cards, for example 2 and 3. They then arrange them side by side to make a two-digit numeral, for example, 23. They then have to make all the possible representations of this with the materials provided, as illustrated in Figure 1.13. (Because the 'teens' do not fit the same language pattern as the subsequent numbers they should be excluded from this activity – teachers should therefore arrange the cards so that one is not used in the tens position for this activity.)

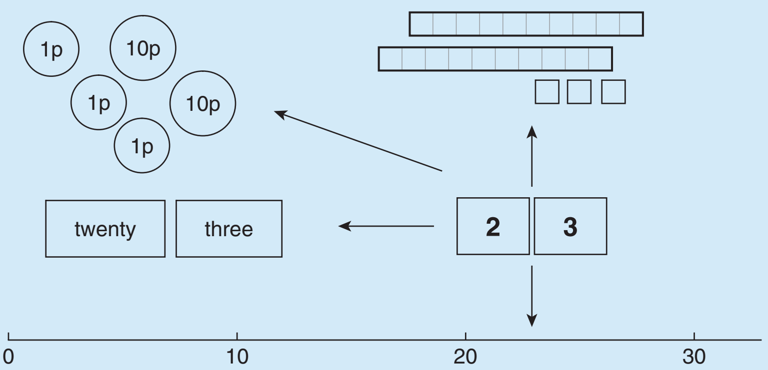

Figure 1.13 *Making place value connections*

Summary of key ideas

1. A simple model for talking about understanding is that to understand something is to connect it with previous learning or other experiences.
2. Mathematical activity involves the manipulation of concrete materials, symbols, language and pictures.
3. Connections between these four types of experience constitute important components of mathematical understanding.
4. A mathematical concept can be thought of as a network of connections between symbols, language, concrete experiences and pictures.
5. A mathematical symbol is a way of representing a mathematical concept that enables us to manipulate it and to discover and express relationships with other concepts.
6. Understanding the concept of place value includes being able to move between the language and symbols used for numbers, concrete experiences with base-ten materials, and the number line.
7. Equivalence and transformation refer respectively to statements of sameness and statements of difference or change.
8. The equals sign strictly means 'is the same as' or 'is equivalent to'. But often in practice it represents both an instruction to apply a transformation – in which case language such as 'makes' and 'leaves' is appropriate – and the equivalence that emerges.
9. One mathematical symbol, such as the equals sign, can be connected with a wide variety of different concrete situations, language and pictures.

Suggestions for further reading

The entries on *Making Connections, Concept Learning, Rote Learning* and *Meaningful Learning* in Haylock with Thangata (2007) are relevant to the discussion of understanding mathematics in this chapter.

Liebeck (1990) uses the same model of understanding as we do in this chapter, but, in contrast to our proposal, she advocates building up connections through sequential experience.

Chapter 3 of Anghileri (2000) provides a useful account of young children's growing understanding of the use of mathematical symbols.

Turner and McCullough (2004) emphasize teaching strategies that seek to establish relationships between language, symbolic notation and pictorial representation.

Gifford (2005) will help those who teach mathematics in the Foundation Stage both to have a more confident understanding of the mathematics they teach and to evaluate their teaching approaches.

The section on mathematical development in the guidance provided for the Foundation Stage Curriculum (QCA, 2000) provides a range of examples of how 3- to 5-year-olds can be helped to connect the language of mathematics with their daily experiences in a rich and interesting school environment.

Using a wide range of practical examples, Montague-Smith (2002) demonstrates how, with sensitive and skilled adult intervention, young children can develop their mathematical understanding within a busy nursery environment.

Tucker (2005), basing her advice on research literature, provides a wide range of stimulating and practical suggestions to help young children develop mathematical connections through play.

A number of chapters in Cockburn and Littler (2008) explore some of the difficulties children encounter with place value in an interesting and accessible way.

Chapter 2 of Haylock (2006) provides a detailed explanation of place value and how this can be understood through making connections.